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The Moyal bracket in the coherent states framework

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Abstract

The star product and Moyal bracket are introduced using the coherent states corresponding to quantum systems with non-linear spectra. Two kinds of coherent state are considered. The first kind is the set of Gazeau–Klauder coherent states and the second kind are constructed following the Perelomov–Klauder approach. The particular case of the harmonic oscillator is also discussed.

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1. Introduction

In classical mechanics, observables are smooth functions on a phase space, which constitute a Poisson algebra, while in quantum mechanics, the observables constitute a non-commutative associative algebra. Deformation quantization is the basis of one of the important attempts aiming to construct a quantum system starting from a classical mechanics system. It is required that the quantum system obtained must go over into the original classical one in the limit $\hbar \to 0$, where $\hbar$ is Planck's constant. In recent times, a deformation quantification has been explored in several contexts: in the string theory approach to non-commutative geometry [1], matrix models [2], the non-commutative Yang–Mills theories [3] and non-commutative gauge theories [4].

Recently, the star product associated with an arbitrary two-dimensional Poisson structure, using the coherent states on the complex plane, was introduced [5]. It was shown that from the coherent states adapted to the harmonic oscillator, one can recover easily the well-known Moyal star product [6]. The deformed coherent states (à la Man'ko *et al*) [7] were also considered to provide an associative star product. Thus, it is clear now that the coherent states formulation gives a useful scheme for defining the star product in a consistent way.

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The approach taken in this work is along the lines of Berezin quantization [8] and relies on coherent (Gazeau–Klauder (GK) [9] and Perelomov–Klauder (PK) [10, 11]) states adapted to an exactly solvable system with a non-linear spectrum [12, 13]. The use of the coherent states is prompted by their useful property of overcompleteness. For our purpose, we will consider the coherent states à la GK and one defined following the PK approach. These constructions lead us, as we will see, to inequivalent states except for the harmonic oscillator case.

We start by introducing the creation and annihilation operators corresponding to a quantum system with non-linear spectra of the type $e_n = an^2 + bn$ ($n \in \mathbb{N}, a \geqslant 0, b > 0$). For some particular values of a and b, one finds again well-known quantum mechanical systems like the Pöschl–Teller potential (see [12]), the x^4-anharmonic oscillator [14] and the standard harmonic oscillator. Section 3 is devoted to the construction of GK and PK coherent states for the above non-linear quantum systems. Their properties (resolution to unity and analytical representations) are also presented. In section 4, the GK coherent states (eigenstates of the annihilation operator) lead easily to the definition of the star product and Moyal bracket on the complex plane. However, when one deals with PK coherent states, the previous definition becomes non-trivial due to the fact that coherent states of this kind are not eigenstates of the annihilation operator. To overcome this difficulty, we introduce a new annihilation operator that diagonalizes the PK states. Concluding remarks are given in the last section.

2. Non-linear quantum spectra

Choose a Hamiltonian H with a discrete spectrum which is bounded below, and has been adjusted such that $H \geqslant 0$. We assume that the eigenvalues of H are non-degenerate. The eigenstates $|\psi_n\rangle$ of H are orthonormal vectors and they satisfy

$$H|\psi_n\rangle = e_n|\psi_n\rangle. \tag{1}$$

We suppose that the $e_n \geqslant 0$, and are such that $e_{n+1} > e_n$. The energy e_0 of the ground state $|\psi_0\rangle$ is chosen to be zero. It is well known that for such a system one can factorize the Hamiltonian H in terms of creation (a^+) and annihilation (a^-) operators as follows:

$$H = a^+a^-. \tag{2}$$

These operators act on the Hilbert space $\mathcal{H} = \{|\psi_n\rangle, n \in \mathbb{N}\}$ as

$$a^+|\psi_n\rangle = \sqrt{e_{n+1}}|\psi_{n+1}\rangle \qquad \text{and} \qquad a^-|\psi_n\rangle = \sqrt{e_n}|\psi_{n-1}\rangle, \tag{3}$$

implemented by $a^-|\psi_0\rangle = 0$. We define the operator G as

$$[a^-, a^+] = G \sim G(N), \tag{4}$$

the commutator between a^- and a^+ . It is clear, from equation (3), that the action of G on the state $|\psi_n\rangle$ is given by

$$G|\psi_n\rangle = [a^-, a^+]|\psi_n\rangle = (e_{n+1} - e_n)|\psi_n\rangle. \tag{5}$$

The operator N is defined such that

$$N|\psi_n\rangle = n|\psi_n\rangle. \tag{6}$$

Note that, in general, the operator N is different from H. It coincides only in the harmonic oscillator case. Furthermore, one can verify also the following commutation relations:

$$[N, a^\pm] = \pm a^\pm. \tag{7}$$

In this article, as we have mentioned before, we focus our attention on quantum systems with energy spectra of the type

$$e_n = an^2 + bn, \qquad n = 0, 1, 2, \ldots, \; a \geqslant 0, \; b > 0. \tag{8}$$

This choice covers many interesting situations. Indeed, for $(a = 1, b = k + k')$, we have the spectra of a quantum system evolving in the Pöschl–Teller potentials parametrized by k and k' ($k > 1$ and $k' > 1$) [12, 13]:

$$H = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x), \tag{9}$$

where

$$V(x) = \frac{1}{4}\left(\frac{k(k-1)}{\sin^2(x/2)} + \frac{k'(k'-1)}{\cos^2(x/2)}\right) \qquad 0 < x < \pi, \tag{10}$$

and $V(x) = \infty$ otherwise (i.e. $x \geqslant 0$; $x \geqslant \pi$). This family of potentials is also called, sometimes, the Pöschl–Teller potentials of the first kind. The latter can be reduced to other interesting potentials, which are widely used in solid-state and molecular physics, like the Scarf and Rosen–Morse ones (see [12] and references quoted therein). The case $(a = 1, b = k + k' = 2)$ corresponds to the spectra of a free particle trapped in an infinite square-well potential. In the case $(a = 3\epsilon/2, b = a + 1)$, where the parameter ϵ is positive, we have the energy levels of the so-called x^4-anharmonic oscillator [14] described by the Hamiltonian

$$H = a_0^+ a_0^- + \frac{\epsilon}{4}(a_0^- + a_0^+)^4 - c_0, \tag{11}$$

where $c_0 = 3\epsilon/4 - 21\epsilon^2/2$ and a_0^+, a_0^- are annihilation and creation operators ($\{a_0^-, a_0^+\} = 1$) for the harmonic oscillator. This quantum system has been extensively studied since the early 1970s (see review [14]). Finally, for $a = 0$ and $b = 1$, we obtain the standard harmonic oscillator spectra which can be also obtained from the x^4-anharmonic system in the limit $\epsilon \to 0$.

3. Coherent states

Coherent states play an important role in many different contexts of theoretical and experimental physics, especially quantum optics [11]. This notion was first discovered for the harmonic oscillator and has been extended to several other potentials in the references [9, 12, 13] in which the coherent states are defined: (i) as eigenstates of the annihilation operator, (ii) by acting with the displacement operator on the ground state $|\psi_0\rangle$ and (iii) as states minimizing the so-called Robertson–Shrödinger uncertainty relations. The definitions (i), (ii), (iii) give different sets of states when one deals with a quantum system other than the harmonic oscillator. As we have mentioned above, we investigate the way to construct the star product using the coherent states associated with an arbitrary quantum system having spectra of the type $e_n = an^2 + bn$. Two types of coherent state will be used. The first set is the so-called GK coherent states obtained from the definition (i). The second type is the PK constructed following the definition (ii). Note that the minimization of the Robertson–Shrödinger uncertainty relation leads to the so-called generalized intelligent states, which are not of interest in this work. In what follows, we briefly review the relevant definitions and results regarding the coherent states.

3.1. Gazeau–Klauder coherent states

Let us denote the GK coherent states by $|z\rangle$, $z \in \mathbb{C}$. They are defined as eigenstates of the annihilation operator a^-:

$$a^-|z\rangle = z|z\rangle. \tag{12}$$

Decomposing $|z\rangle$ in the Hilbert space $(\mathcal{H})$ basis, and using the action of a^- on the $|\psi_n\rangle$ states given by (3), we show that the coherent states $|z\rangle$ are as follows:

$$|z\rangle = (\mathcal{N}(|z|^2))^{-1} \sum_{n=0}^{\infty} \frac{(\Gamma(r+1))^{1/2} z^n}{(n!\Gamma(n+r+1))^{1/2} a^{n/2}} |\psi_n\rangle, \tag{13}$$

where $r = b/a$ and $\mathcal{N}(|z|^2)$ the normalization constant which is defined by

$$\mathcal{N}(|z|^2) = \left({}_0F_1\left(r+1, \frac{|z|^2}{a}\right)\right)^{1/2}. \tag{14}$$

The set of states $|z\rangle$ is overcomplete. Indeed, the resolution of unity:

$$\int |z\rangle\langle\bar{z}| \, \mathrm{d}\mu(z,\bar{z}) = I_{\mathcal{H}}, \tag{15}$$

is ensured with respect to the measure

$$\mathrm{d}\mu(z,\bar{z}) = \frac{2}{\pi a} I_r\left(\frac{2r}{\sqrt{a}}\right) K_{r/2}\left(\frac{2r}{\sqrt{a}}\right) r \, \mathrm{d}r \, \mathrm{d}\theta, \qquad z = r\mathrm{e}^{\mathrm{i}\theta}. \tag{16}$$

The latter formula can be determined in different ways. Here, we have used the approach developed in [12, 13]. The kernel (overlapping of two coherent states) is given by

$$\langle z'|z\rangle = \frac{{}_0F_1(r+1, \bar{z'}z/a)}{({}_0F_1(r+1, |z|^2/a)\,{}_0F_1(r+1, |z'|^2/a))^2}. \tag{17}$$

The overcompletion of the set $\{|z\rangle, z \in \mathbb{C}\}$ provides a representation of any state $|f\rangle$ by the entire function

$$f(z) = \left({}_0F_1\left(r+1, \frac{|z|^2}{a}\right)\right)^{1/2} \langle\bar{z}|f\rangle, \tag{18}$$

In particular, the analytic functions corresponding to the eigenstates $|\psi_n\rangle$ are

$$\mathcal{F}_n(z) = \frac{z^n\sqrt{\Gamma(r+1)}}{a^{n/2}(n!\Gamma(n+r+1))^{1/2}}. \tag{19}$$

On the set $\{\mathcal{F}_n(z)\}$, the actions of the creation and annihilation operators are given by

$$a^+ = z, \qquad a^- = \left(z\frac{\mathrm{d}^2}{\mathrm{d}z^2} + (r+1)\right)\frac{\mathrm{d}}{\mathrm{d}z}, \tag{20}$$

and the operator G acts as

$$G = 2az\frac{\mathrm{d}}{\mathrm{d}z} + (a+b). \tag{21}$$

It is easy to see that these operators act in the functions space, $\{\mathcal{F}_n(z), n \in \mathbb{N}\}$, as

$$a^+\mathcal{F}_n(z) = \sqrt{e_{n+1}}\mathcal{F}_{n+1}(z) \qquad a^-\mathcal{F}_n(z) = \sqrt{e_n}\mathcal{F}_{n-1}(z) \qquad G\mathcal{F}_n(z) = (e_{n+1} - e_n)\mathcal{F}_n(z). \tag{22}$$

This realization will be useful in the remainder of this work when we introduce the star-product approach based on the GK coherent states.

3.2. Perelomov–Klauder coherent states

We recall that the PK coherent states are defined by

$$|z\rangle = \mathcal{D}(z)|\psi_0\rangle = \exp(za^+ - \bar{z}a^-)|\psi_0\rangle. \tag{23}$$

The computation of the action of the displacement operator $\mathcal{D}(z)$ on the ground state $|\psi_0\rangle$ was done for an arbitrary quantum system and illustrated for the Pöschl–Teller potentials [13]. Note that this result can also be applied for a quantum system possessing energy levels $e_n = an^2 + bn$ ($n \in \mathbb{N}$) with a minor modification. Then, one can obtain

$$|\zeta\rangle = (1 - |\zeta|^2)^{(r+1)/2} \sum_{n=0}^{\infty} \sqrt{\frac{\Gamma(n+r+1)}{n!\Gamma(r+1)}} \zeta^n |\psi_n\rangle, \tag{24}$$

where $\zeta = (z/|z|)\tanh(z\sqrt{a})$. The states $|\zeta\rangle$ satisfy the resolution to unity, namely

$$\int |\zeta\rangle\langle\zeta| \, \mathrm{d}\mu(\zeta, \bar{\zeta}) = I_{\mathcal{H}}, \tag{25}$$

with respect to the measure given by

$$\mathrm{d}\mu(\zeta, \bar{\zeta}) = \frac{r}{\pi} \frac{\mathrm{d}^2\zeta}{(1 - |\zeta|^2)^2}. \tag{26}$$

The kernel $\langle\zeta'|\zeta\rangle$ is given by

$$\langle\zeta'|\zeta\rangle = (1 - |\zeta'|^2)^{(r+1)/2} (1 - |\zeta|^2)^{(r+1)/2} \sum_{n=0}^{\infty} \frac{\Gamma(n+r+1)}{n!\Gamma(r+1)} (\bar{\zeta}'\zeta)^n. \tag{27}$$

The state $|\psi_n\rangle$ is represented analytically by the function

$$\mathcal{G}_n(\zeta) = \zeta^n \sqrt{\frac{\Gamma(n+r+1)}{n!\Gamma(r+1)}}. \tag{28}$$

The creation and annihilation operators act in the Hilbert space of analytical functions $\{\mathcal{G}_n(\zeta), n \in \mathbb{N}\}$ as first-order differential operators:

$$a^+ = \zeta^2 \frac{\mathrm{d}}{\mathrm{d}\zeta} + (r+1)\zeta, \qquad a^- = \frac{\mathrm{d}}{\mathrm{d}\zeta}, \tag{29}$$

and the operator G acts in the same representation as

$$G = 2\zeta \frac{\mathrm{d}}{\mathrm{d}\zeta} + (r+1). \tag{30}$$

One can verify that

$$a^+\mathcal{G}_n(\zeta) = \sqrt{e_{n+1}}\mathcal{G}_{n+1}(\zeta) \qquad a^-\mathcal{G}_n(\zeta) = \sqrt{e_n}\mathcal{G}_{n-1}(\zeta) \qquad G\mathcal{G}_n(\zeta) = e_n\mathcal{G}_n(\zeta). \tag{31}$$

To end this subsection, we would like to note that the analytical representations of both the GK and PK coherent states are related through the Laplace transform [15].

4. The star product and Moyal bracket

In this section, we introduce the star product and Moyal bracket in the coherent states framework. Let us start by recalling the definition of the star product. With every operator A acting on the Hilbert space $\mathcal{H}$, one can associate a function $\mathcal{A}(z, \bar{z})$ on the complex plane as

$$\mathcal{A}(z, \bar{z}) = \langle z|A|z\rangle \tag{32}$$

The associative star product of the two functions $\mathcal{A}(z,\bar{z})$ and $\mathcal{B}(z,\bar{z})$ is defined in [5] by

$$\mathcal{A}(z,\bar{z}) \star \mathcal{B}(z,\bar{z}) = \langle z|AB|z\rangle, \tag{33}$$

and then the corresponding Moyal bracket is given by

$$\{\mathcal{A}(z,\bar{z}),\mathcal{B}(z,\bar{z})\}_M = \mathcal{A}(z,\bar{z}) \star \mathcal{B}(z,\bar{z}) - \mathcal{B}(z,\bar{z}) \star \mathcal{A}(z,\bar{z}) = \langle z|[A,B]|z\rangle. \tag{34}$$

Using the identity resolution of the coherent states, the star product equation (33) becomes

$$\mathcal{A}(z,\bar{z}) \star \mathcal{B}(z,\bar{z}) = \int \mathrm{d}\mu(\zeta,\bar{\zeta})\, \langle z|A|\zeta\rangle\langle\zeta|B|z\rangle, \tag{35}$$

which can also be written as

$$\mathcal{A}(z,\bar{z}) \star \mathcal{B}(z,\bar{z}) = \sum_{n,m} \langle z|\psi_n\rangle\langle\psi_n|AB|\psi_m\rangle\langle\psi_m|z\rangle, \tag{36}$$

in terms of the functions $\langle\psi_m|z\rangle$ corresponding to the element $|\psi_m\rangle$ of the Hilbert space $\mathcal{H}$. It follows that the Moyal bracket takes the form

$$\{\mathcal{A}(z,\bar{z}),\mathcal{B}(z,\bar{z})\}_M = \sum_{n,m} \langle z|\psi_n\rangle\langle\psi_n|[A,B]|\psi_m\rangle\langle\psi_m|z\rangle. \tag{37}$$

Analysing the relations (37), we see that there is a correspondence between the structure relations of the operator algebra acting on the Hilbert space and the star commutators, namely the Moyal bracket, of the elements generating the algebra of the functions on the complex plane. This point will be examined throughout this section.

We note also that the star product (33) can be written in the integral representation in terms of the ordered exponential [5]:

$$\star = \int \mathrm{d}\mu(\zeta,\bar{\zeta}) : \exp\left(\frac{\overrightarrow{\partial}}{\partial\eta}(\zeta-\eta)\right) : |\langle\eta|\zeta\rangle|^2 : \exp\left((\bar{\zeta}-\bar{\eta})\frac{\overleftarrow{\partial}}{\partial\bar{\eta}}\right), \tag{38}$$

which is not used in this work.

4.1. The star product with Gazeau–Klauder coherent states

In the GK coherent states, the star product will take the simple form

$$\mathcal{A}(z,\bar{z}) \star \mathcal{B}(z,\bar{z}) = \mathcal{N}(|z|^2)^{-2} \sum_{n,m} \mathcal{F}_n(\bar{z})\langle\psi_n|AB|\psi_m\rangle\mathcal{F}_m(z). \tag{39}$$

Since the GK coherent states are the eigenstates of the annihilation operators a^-, there is a correspondence between a^- and the analytic function $z \to z$:

$$\langle z|a^-|z\rangle = z. \tag{40}$$

Then, the anti-analytic function $z \to \bar{z}$ is the expectation value of the operators a^+ over the coherent state $|z\rangle$:

$$\langle z|a^+|z\rangle = \bar{z}. \tag{41}$$

Furthermore, by using the definition of the star product (39) one can obtain easily the following relations:

$$1 \star 1 = 1 \qquad 1 \star z = z \star 1 = z \qquad 1 \star \bar{z} = \bar{z} \star 1 = \bar{z}, \tag{42}$$

and, more generally, we have

$$z^{\star p} = z^p \qquad \bar{z}^{\star p} = \bar{z}^p \qquad p \geqslant 0, \tag{43}$$

where $\theta^{\star p} = \theta \star \theta \star \theta \ldots \star \theta$ (p times, with $\theta = z$ or $\bar{z}$). On the basis of the latter relations, one can evaluate the star product for any z-analytic and $\bar{z}$-anti-analytic functions which are given by

$$\mathcal{A}(z) \star \mathcal{B}(z) = \mathcal{A}(z)\mathcal{B}(z), \tag{44}$$

and

$$\mathcal{A}(\bar{z}) \star \mathcal{B}(\bar{z}) = \mathcal{A}(\bar{z})\mathcal{B}(\bar{z}). \tag{45}$$

We show also that

$$\bar{z} \star z = \langle z|a^+a^-|z\rangle = z\bar{z} = |z|^2. \tag{46}$$

Hence the star product of two functions is reduced to the ordinary one if the function on the right is analytic and the function on the left is anti-analytic:

$$\mathcal{A}(\bar{z}) \star \mathcal{B}(z) = \mathcal{A}(\bar{z})\mathcal{B}(z). \tag{47}$$

For completeness, we will compute the star product of type $z \star \bar{z}$. From the previous considerations, it is easy to see that

$$z \star \bar{z} = \langle z|a^-a^+|z\rangle = \bar{z} \star z - \mathcal{G}(z, \bar{z}), \tag{48}$$

where

$$\mathcal{G}(z, \bar{z}) = \frac{2a|z|^2}{r+1} \frac{{}_0F_1(r+2, |z|^2/a)}{{}_0F_1(r+1, |z|^2/a)} + (a+b). \tag{49}$$

The function $\mathcal{G}(z, \bar{z})$ can be expressed as follows:

$$\mathcal{G}(z, \bar{z}) = 2az\frac{\mathrm{d}}{\mathrm{d}z} \sum_n \mathcal{F}_n(\bar{z})\mathcal{F}_n(z) + a + b, \tag{50}$$

in terms of the functions $\{\mathcal{F}_n(z), n \in \mathbb{N}\}$.

One remarks that the Moyal bracket preserves the commutation relations of the algebra generated by $\{a^-, a^+, G\}$ as we have already mentioned. Indeed, in the operator language we have the following relations:

$$[a^-, a^+] = G(N), \qquad [a^\pm, G(N)] = \pm 2aa^\pm, \tag{51}$$

which are expressed in the language of the Moyal bracket as

$$\{z, \bar{z}\}_M = \mathcal{G}(z, \bar{z}), \qquad \{z, \mathcal{G}(z, \bar{z})\}_M = 2az, \qquad \{\bar{z}, \mathcal{G}(z, \bar{z})\}_M = -2a\bar{z}. \tag{52}$$

Using the previous results, one can show the following interesting relations:

$$\begin{aligned}
\bar{z} \star \mathcal{G}(z, \bar{z}) &= \mathcal{G}(z, \bar{z}) + \bar{z}(a+b) - (a+b),\\
\mathcal{G}(z, \bar{z}) \star \bar{z} &= \mathcal{G}(z, \bar{z}) + \bar{z}(3a+b) - (a+b),\\
\mathcal{G}(z, \bar{z}) \star z &= \mathcal{G}(z, \bar{z}) + z(a+b) - (a+b),\\
z \star \mathcal{G}(z, \bar{z}) &= \mathcal{G}(z, \bar{z}) + z(3a+b) - (a+b),
\end{aligned} \tag{53}$$

which are useful for calculating in a complete way the star product. Finally, using the relations (42), (43), (46), (52) and (53), one can compute the star product of any two functions $\mathcal{A}(z, \bar{z})$ and $\mathcal{B}(z, \bar{z})$. As an illustration, let us give the following example: the star products of $\mathcal{A}(z, \bar{z}) = \bar{z}$ and $\mathcal{B}(z, \bar{z}) = \bar{z}z$ are given by

$$\begin{aligned}
\mathcal{A}(z, \bar{z}) \star \mathcal{B}(z, \bar{z}) &= \bar{z}^2 z,\\
\mathcal{B}(z, \bar{z}) \star \mathcal{A}(z, \bar{z}) &= \bar{z}^2 z + \mathcal{G}(z, \bar{z}) + \bar{z}(a+b) - (a+b),
\end{aligned} \tag{54}$$

and the corresponding Moyal bracket is defined as

$$\{\mathcal{A}(z, \bar{z}), \mathcal{B}(z, \bar{z})\}_M = (a+b) - \mathcal{G}(z, \bar{z}) - \bar{z}(a+b). \tag{55}$$

In this particular case: $a = 0$ and $b = 1$ (i.e., the harmonic oscillator case), the function $\mathcal{G}(z, \bar{z})$ is equal to unity and relations (53) reduce to relations (42). The relation (52) gives the well-known Moyal bracket, constructed using the coherent states adapted to the standard harmonic oscillator [5].

It is true that for the quantum systems considered in this work, we can define the GK coherent states as well as the PK ones. However, it should be noted that there exist some quantum systems for which the GK coherent states cannot be constructed, due to the fact that the dimension of the Hilbert space $\mathcal{H}$ is finite like a quantum system trapped in the Morse potential [16]. In this situation, the definition of the star product discussed above cannot be used. So, for this reason, we believe that it is interesting to introduce also the star product in the PK coherent states, which will be the focus of the following section.

4.2. The star product with Perelomov–Klauder coherent states

The PK coherent states $|\zeta\rangle$ equation (24) are not the eigenstates of the annihilation operator a^-. In order to define the star product and Moyal bracket, one may ask whether the analytic function $\zeta \rightarrow \zeta$ and the anti-analytic function $\zeta \rightarrow \bar{\zeta}$ can be defined as the mean values of some operators A^- and A^+ acting on the Hilbert space $\mathcal{H}$:

$$\langle\zeta|A^-|\zeta\rangle = \zeta \qquad \langle\zeta|A^+|\zeta\rangle = \bar{\zeta}. \tag{56}$$

Let us introduce the operators

$$A^- = a^- f(N) \qquad A^+ = f(N)a^+. \tag{57}$$

The operators A^- and A^+ satisfy the relations (56), when $f(N)$ is defined by

$$f(N) = \frac{N+1}{g(N+1)}, \tag{58}$$

where the function operator $g(N+l)$ acts in the Hilbert space $\{|\psi_n\rangle, n \in \mathbb{N}\}$ as

$$g(N+l)|\psi_n\rangle = e_{n+l}|\psi_n\rangle, \tag{59}$$

for $l \in \mathbb{N}$. The new operators A^- and A^+ satisfy the following relations:

$$[A^-, A^+] = D(N), \tag{60}$$

where the operator $D(N)$ is defined as a function of the operator N by

$$D(N) = \frac{(N+1)^2}{g(N+1)} - \frac{N^2}{g(N)}. \tag{61}$$

One can also show that

$$A^- D(N) = D(N+1)A^- \qquad A^+ D(N) = D(N-1)A^+. \tag{62}$$

For our purposes, we define the following functions:

$$\mathcal{D}_l(\zeta, \bar{\zeta}) = \langle\zeta|D(N+l)|\zeta\rangle, \tag{63}$$

which are useful in the computation of the star product based on the PK coherent states. A straightforward calculation leads to

$$\mathcal{D}_l(\zeta, \bar{\zeta}) = (1 - |\zeta|^2)^{r+1} \sum_n \mathcal{G}_n(\bar{\zeta})\mathcal{G}_n(\zeta)\left\{(n+l+2)^2 \frac{e_{n+l+1}}{e_{n+l+2}^2} - (n+l+1)^2 \frac{e_{n+l}}{e_{n+l+1}^2}\right\}. \tag{64}$$

Using (64), one finds the basic relations needed for a computation of a star product between any two functions $\mathcal{A}_l(\zeta, \bar{\zeta})$ and $\mathcal{B}_l(\zeta, \bar{\zeta})$. They are given as

$$\begin{array}{ll} 1 \star \zeta = \zeta \star 1 = \zeta & 1 \star \bar{\zeta} = \bar{\zeta} \star 1 = \bar{\zeta}, \\ \bar{\zeta} \star \zeta = \bar{\zeta}\zeta & \zeta \star \bar{\zeta} = \bar{\zeta}\zeta + \mathcal{D}_0(\zeta, \bar{\zeta}), \\ \zeta \star \mathcal{D}_l(\zeta, \bar{\zeta}) = \zeta \mathcal{D}_{l+1}(\zeta, \bar{\zeta}) & \bar{\zeta} \star \mathcal{D}_l(\zeta, \bar{\zeta}) = \bar{\zeta} \mathcal{D}_l(\zeta, \bar{\zeta}), \\ \mathcal{D}_l(\zeta, \bar{\zeta}) \star \zeta = \zeta \mathcal{D}_l(\zeta, \bar{\zeta}) & \mathcal{D}_l(\zeta, \bar{\zeta}) \star \bar{\zeta} = \bar{\zeta} \mathcal{D}_{l+1}(\zeta, \bar{\zeta}). \end{array} \tag{65}$$

As an application, we set $\mathcal{A}(\zeta, \bar{\zeta}) = \bar{\zeta}$ and $\mathcal{B}(\zeta, \bar{\zeta}) = \bar{\zeta}\zeta$. The star product in this case is given by

$$\begin{array}{l} \mathcal{A}(\zeta, \bar{\zeta}) \star \mathcal{B}(\zeta, \bar{\zeta}) = \bar{\zeta}^2 \zeta, \\ \mathcal{B}(\zeta, \bar{\zeta}) \star \mathcal{A}(\zeta, \bar{\zeta}) = \bar{\zeta}^2 \zeta + \bar{\zeta} \mathcal{D}_0(\zeta, \bar{\zeta}), \end{array} \tag{66}$$

where $\mathcal{D}_0(\zeta, \bar{\zeta})$ is defined by (64).

Contrary to the previous case—the one corresponding to GK coherent states—the structure relations of the algebra $\{a^+, a^-, G\}$ are not preserved by this star product. However, one sees that the Moyal bracket defined from PK coherent states preserves the commutation relations (60) and (62) of the algebra generated by $\{A^+, A^-, D(N)\}$.

For the harmonic oscillator case ($a = 0, b = 1$), we have $f(N) = 1$, $D(N) = 1$ and the operators $A^\pm$ are reduced to the creation and annihilation operators of ordinary harmonic oscillators where the Moyal bracket is trivial.

We conclude that the star product in GK coherent states is in general different from the one obtained in the PK scheme, except for the ordinary oscillator case.

5. Concluding remarks

In this work, we have introduced the star product and the Moyal bracket in the coherent states framework corresponding to exactly solvable quantum systems, admitting non-linear spectra. We have seen that in the PK coherent states case, the construction becomes non-trivial, because the coherent states are not eigenstates of the annihilation operator. This difficulty was removed by introducing a new operator diagonalizing the PK states. The fundamental star products, providing a complete way to compute the Moyal bracket for any two functions, are given in this work. The star product constructed for the standard harmonic oscillator [5] was recorded as a particular case of our approach. It is clear that there remain many problems for future studies. One of them would be the definition of the star product, using the coherent states for the Lie algebras and their supersymmetric counterparts. Another would be a better understanding of the relationship between the star product with GK coherent states and the one using the PK ones. We believe that such a relation can be established, because as we have previously mentioned, the analytical representations of the two types of coherent state are related through Laplace transformation. Finally, it became apparent from this work that the construction of the star product from coherent states implicitly involves a certain rule of correspondence between functions of non-commuting operators and analytical functions; this correspondence is similar to the one between classical and quantum mechanics. So, would be interesting, as suggested by one of the referees of this paper, to show this calculus in use by studying an exactly solvable quantum mechanical system cited in this work (a system trapped in the Pöschl–Teller potential, for instance). This matter is under consideration [17].

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